Analysis I Lecture 3

Last time:

- Talked about axioms of real numbers
- Had some smoke during the lecture.

So How to define reals?

We use the trick:

Instead of constructing $\mathbb{R}$

we list its properties:

1. $\mathbb{Q} \subset \mathbb{R}$
2. $\mathbb{R}$ is an ORDERED FIELD
3. $\mathbb{R}$ satisfies "The INFIMUM AXIOM"

- Field: $\mathbb{R}$ has arithmetic operations

  commutative distributive

  $a+b=b+a$ $a\cdot b=b\cdot a$ $a(b+c)=a\cdot b+a\cdot c$

- Ordered: $\mathbb{R}$ has an order

  $a,b\in\mathbb{R}$ $a<b$ or $a>b$ or $a=b$ +

  compatibility with operations

- Infimum axiom: Every set $S\subset\mathbb{R}_{>0}$

  has the largest $l\in\mathbb{R}$ such that

  $l\leq s$ for every $s\in S$.

Some definitions

Let $S \subset \mathbb{R}$ and $S \neq \emptyset$ then

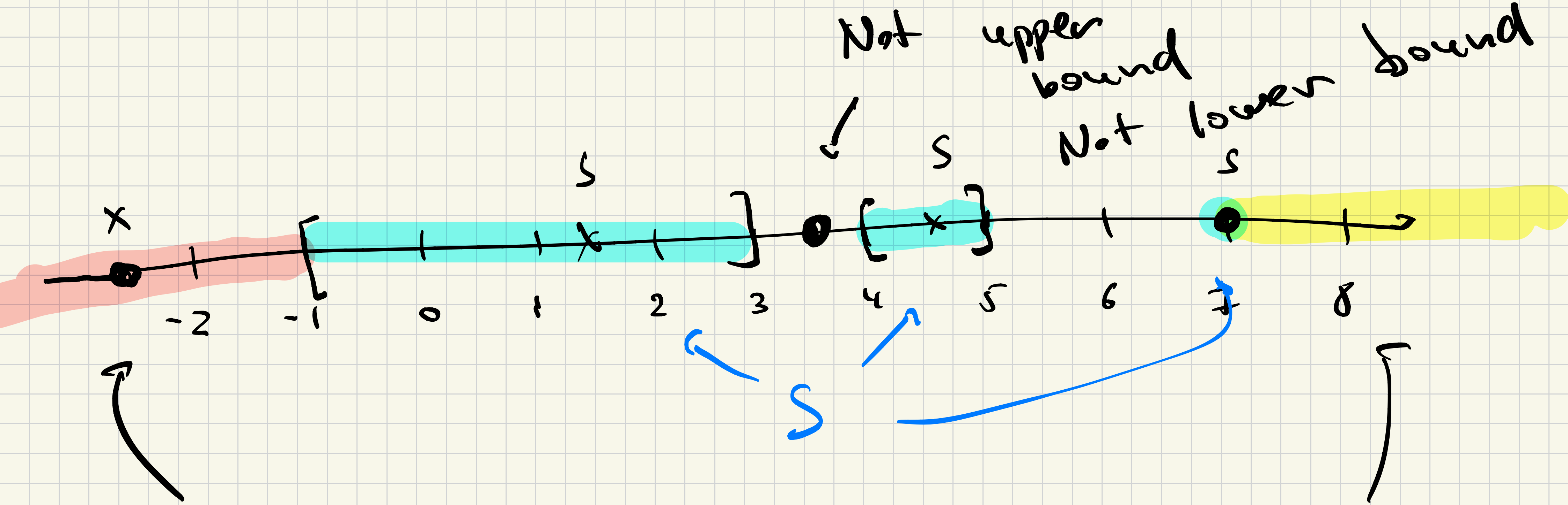
- $a \in \mathbb{R}$ is a lower bound of S
  if $a \leq s$ for any $s \in S$

- $a \in \mathbb{R}$ is an upper bound of S
  if $a \geq s$ for any $s \in S$

Example

- $S = [-1, 3] \cup [4, 5] \cup \{7\}$

$$[-1, 3] = \{ x \mid x \in \mathbb{R} \quad -1 \leq x \leq 3 \}$$



x is a lower bound if and only if $x \leq -1$

x is an upper bound iff $x \geq 7$

Example

$$S = \{ \sin(x) \mid x \in \mathbb{R} \}$$

$$= [-1, 1]$$

Lower bounds: all numbers ≤ -1

Upper bounds: all numbers ≥ 1

Back to definitions

$S \subset \mathbb{R}$, $S \neq \emptyset$

- S is bounded below if it has a lower bound
- S is bounded above if it has an upper bound
- S is bounded if it is bounded both below and above

Example / Notation $\quad a, b \in \mathbb{R} \quad a < b$

- $[a, b]$; $]a, b]$; $[a, b[$; $]a, b[$

$[a,b] = \{x \mid a \leq x \leq b\}$ — Closed interval

$]a, b] = (a, b] = \{x \mid a < x \leq b\}$

$[a, b[= [a, b) = \{x \mid a \leq x < b\}$

$\underbrace{}_{\text{half-closed intervals}}$

half-closed intervals

half-open

$]a, b[= (a, b) = \{x \mid a < x < b\}$ — open interval

- $[a, +\infty)$ $\quad (-\infty, b]$

$[a, b]$, $[a, b)$, $(a, b]$, (a, b)

are bounded (both from below and above)

- Rays: $[a, +\infty) = \{x \mid x \geq a\}$

$(a, +\infty) = \{x \mid x > a\}$

bounded below but not ABOVE

$(-\infty, b] = \{x \mid x \leq b\}$

$(-\infty, b) = \{x \mid x < b\}$

bounded above but not below

Proposition

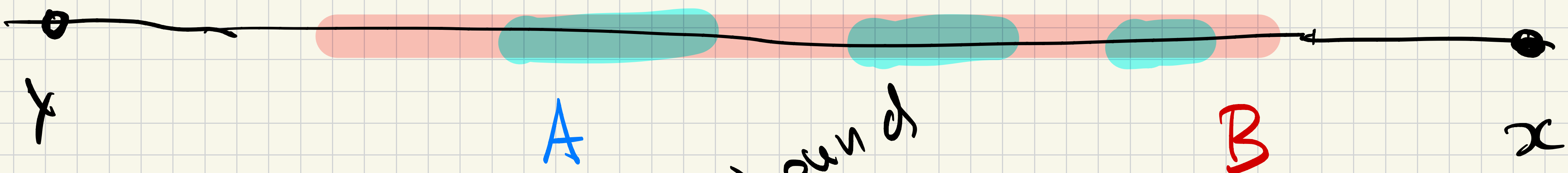
Let $A \subset B \subset \mathbb{R}$, $A \neq \emptyset$ [In particular $B \neq \emptyset$]

if B is bounded below (above)

then A is bounded below (above).

In particular, if B is bounded, then A is bounded

Exercise: to prove this proposition.



if is lower Bound for B then also lower bound for A

upper bound of B. Then it is an upper bound for A,

__Example.__

$\mathbb{Z}$ is not bounded above or below:

For any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ s.t $n > x$

Example.

$\mathbb{Z}$ is not bounded above or below:

For any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ s.t $n > x$

But also there exists $m \in \mathbb{Z}$ s.t $m < x$

$\Rightarrow$ No real number can be an upper or lower bound for $\mathbb{Z}$.

MIN / MAX   INF / SUP

<u>Definition</u> • a is the minimum of S

if $a \in S$ and a is a lower bound.

• a is the maximum of S if

$a \in S$ and a is an upper bound.

<u>Notation:</u> $\min(S)$ $\max(S)$

$\min S$ $\max S$

Example $S = [0, 1) = \{ 0 \leq x < 1 \}$

lower bounds of S are $\{ x \mid x \leq 0 \} = (-\infty, 0]$

Want to find $e \in (-\infty, 0] \cap S = \{0\}$

so $\min(S) = 0$

The set of upper bounds is $[1, +\infty)$

but $[1, +\infty) \cap [0, 1) = \emptyset$ so

S does not have maximum

Proposition $S \subseteq \mathbb{R}$, $S \neq \emptyset$

- If min (max) of S exists, they are unique.
- If S is not bounded above then max (S) does not exists
- If S is not bounded below then min(S) does not exists.

PROOF:

If min (max) of S exists, they are unique.

In other words if x, y are minimums of S then $x = y$.

Both x, y are lower bounds of S

$x \in S$, $y \in S$

$\Rightarrow$ $x \leq y$ since x is lower bound and $y \in S$

$y \leq x$ since y is lower bound and $x \in S$

$$\left.\begin{matrix} x \leq y \\ y \leq x \end{matrix}\right\} \Longrightarrow \quad x = y \quad \blacksquare$$

- If S is not bounded above then $\max(S)$ does not exists

Proof: If $\max(S)$ exists then there is an upper bound of S so S is bounded above. $\blacksquare$

What if no min / max?

Definition $S \subseteq \mathbb{R}$, $S \neq \emptyset$

- $a \in \mathbb{R}$ is the SUPREMUM of S if a is the smallest upper bound.
- $a \in \mathbb{R}$ is the INFIMUM of S if a is the largest lower bound.

Notation: $\inf(S)$, $\sup(S)$.

Does infimum / supremum exist?

Infimum axiom implies that

Any S bounded above has supremum and any S bounded below has infimum.

(Infimum Axiom: Any $S \subset \mathbb{R}_{>0}$ has infimum.)

Proposition

Supremum / Infimum is unique.

Proof: Let x, y be infimums of S bounded below:

x is a lower bound, $x \geq$ any other lower bound

y is a lower bound, $y \geq$ any other lower bound

So we get:

$$x \geq y \quad \text{and} \quad y \geq x \Rightarrow x = y.$$

Same proof works for sup.

$\blacksquare$

Let S be bounded below

then $\min(S)$ exists if and

only if $\inf(S) \in S$ and

in this case

$$\inf(S) = \min(S).$$

Same for sup and max.

Proposition

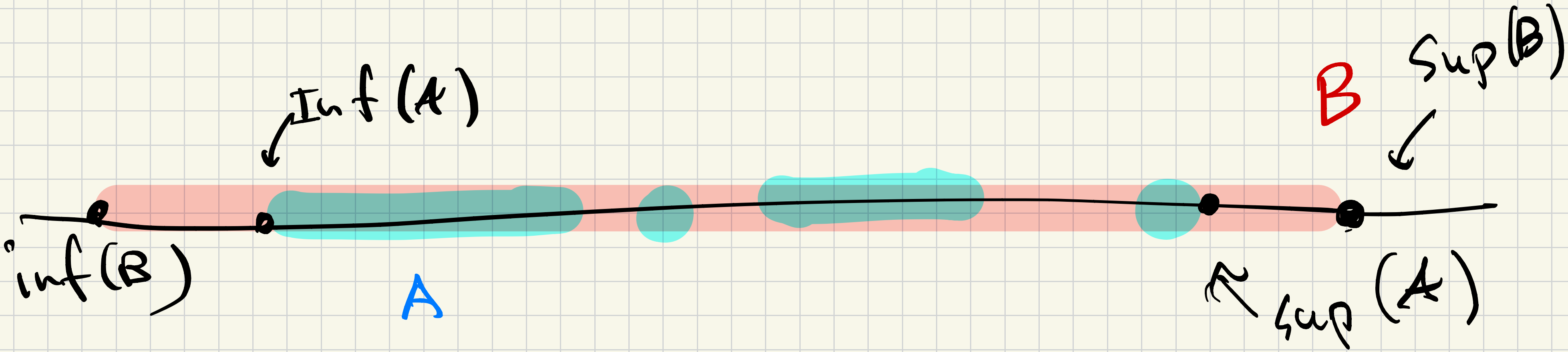
Let $A \subseteq B$ and $A \neq \emptyset$ then

- If B is bounded below then

$$\inf(A) \geq \inf(B)$$

- If B is bounded above then

$$\sup(A) \leq \sup(B)$$



Another definition of INF, sup

Notation:

$\forall$ – for any

$\exists$ – there exists

$\exists!$ – there exists unique

$\not\forall$ = "there is no"

$\nexists$ = "there is not"

Example: x is an upper bound of S:

Example: x is an upper bound of S:

$x \geq s$ for any $s \in S$

$x \geq s \quad \forall s \in S$

equivalently

If there is no $s \in S$ such that s.t.

$x < s$.

$\not\forall s \in S$ such that $x < s$.

Alternative definition of sup:

Theorem $x = \sup(S)$ if and only if

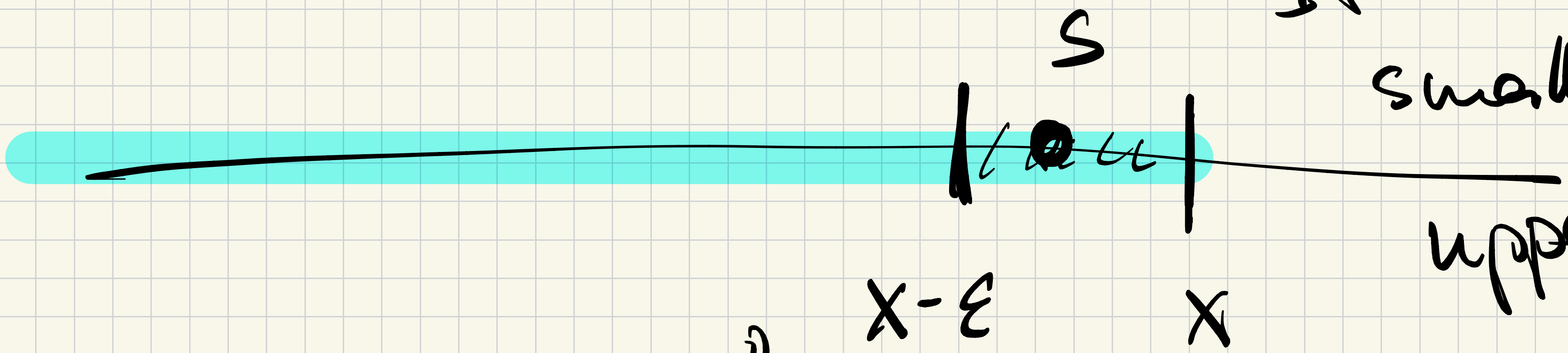
$\underbrace{\forall s \in S, \quad x \geq s}_{\text{x is an upper bound for S}}$ and $\forall \varepsilon > 0 \ \exists s \in S$ s.t. $s \geq x - \varepsilon$.

For any $\varepsilon > 0$ there exists $s \in S$ s.t. $s \geq x - \varepsilon$

<u>Remark</u> To understand the statements of theorems / definitions every time you see $\varepsilon > 0$ You should imagine it to be really small positive number.

Back to the theorem:

If x is smallest upper bound



then $x - \varepsilon$ is not an upper bound

Number slightly smaller than x if $\varepsilon > 0$

$\Rightarrow \exists\, s \in S \quad s > x - \varepsilon$